

Covariant forms of Lax one-field operators: from Abelian to non-commutative

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Abstract

Polynomials in differentiation operators are considered. The Darboux transformations covariance determines non-Abelian entries to form the coefficients of the polynomials. Joint covariance of a pair of such polynomials (Lax pair) as a function of one-field is studied. Methodically, the transforms of the coefficients are equalized to Frechet derivatives (first term of the Taylor series on prolonged space) to establish the operator forms. In the commutative (Abelian) case that results in binary Bell (Faa de Bruno) differential polynomials having natural bilinear representation. The example of generalized Boussinesq equation is studied, the chain equations for the case are derived. A set of integrable non-commutative potentials and hence nonlinear equations is constructed altogether with explicit dressing formulas.

1 Introduction.

Investigations of general DT theory in the case of differential operators

$$L = \sum_{k=0}^n a_k \partial^k \quad (1.1)$$

with non-commutative coefficients was launched by papers of Matveev [1]. The proof of a general covariance of the equation

$$\psi_t = L\psi \quad (1.2)$$

with respect to the classic Darboux transformation

$$\psi[1] = \psi' - \sigma\psi \quad (1.3)$$

incorporates the auxiliary relation

$$\begin{aligned} \sigma_t &= \partial r + [r, \sigma], \\ r &= \sum_0^N a_n B_n(\sigma), \end{aligned} \quad (1.4)$$

where B_n are differential Bell (Faa de Bruno [5]) polynomials [6]. The relation (1.4) generalize so-called Miura map and became the identity when $\sigma = \phi' \phi^{-1}$, ϕ is a solution of the equation (4.52).

Such combinations may be used in the Lax representation constructions for nonlinear problems. It opens the way to produce wide classes of solutions of the nonlinear problem. This known approach [7] intensely develops nowadays. Examples of non-Abelian and non-local equations, integrable by DT was considered in [2], [4]. Some of them were reviewed and developed in the book [7]. The general approach was developed for difference operator (see again the papers cited above) and recently generalized for wide class of polynomials of automorphism on a differential ring [10].

The choice of the jointly covariant combinations introduces addition problems of the appropriate choice of potentials on which the polynomial coefficients depend [15]. This problem was recently discussed in [12], where a method of the conditions account was developed. Covariant combinations of (generalized) derivatives and potentials may be hence classified for linear problems. The task is intimately connected with other reduction problems of Lax operator representation of nonlinear equations. In two words, having the general statement about covariant form of a linear polynomial differential operator that determine transformation formulas for coefficients (Darboux theorem and its Matveev's generalizations), the consistency between two such formulas need the special definition of potentials. For example, the second order scalar differential operator has the only place for a potential and the covariance generate the classic Darboux transformation for it. The place that this potential take in the second equation of the KdV Lax pair needs special investigation.

In scalar case such one-potential construction have been studied in [13] and developed for higher KdV and KP equations [14]. It was found that the result is conveniently written via such combinations of differentiation operator and exponential functions of the potential as Binary Bell Polynomials (BBP) [11]. The principle and result is reproduced and developed in the Sec. 1.2 of this paper to give more explanation.

The whole construction in general (nonAbelian) case could be similar, a bit more complicated, but much more rich and promising. The theory could contain two ingredients.

i) The first one would be non-Abelian Hirota construction in the terms of the mentioned binary Bell polynomials [11]. On the level of general expressions and applications some obstacle appears, e.g. an extension of addition formulas [11]) to the nonAbelian case.

ii) The second way relates to some generalized polynomials that could be produced as covariant combinations of operators with a faith that observations from Abelian theory could be generalized. Namely the case we would discuss in this paper.

Even for the minimal (first order in the D-operator) examples one arrives to the operator ZS problems that contain many interesting integrable models. It is seen already from the point of view of symmetry classification [9]. So, the link to DT covariance approach allows to hope for a realization of the main purpose - construction of covariant polynomials and their enumeration and use in the soliton equations theory.

We would begin from the example, using notations from quantum mechanics to emphasize the non-Abelian nature of the consideration. The operators ρ and H could play the roles of density matrices and Hamiltonians, respectively, but one also can think of them as just some operators without any particular quantum mechanical connotations. The approach establishes the covariance with respect to bDT [?] of rather general Lax system for the equation

$$-i\rho_t = [H, h(\rho)],$$

where $h(\rho)$ - analytic function, in some sense -"Abelian", the function to be defined by Taylor series [20]. More exactly it is shown that the following statement takes place

Theorem. Assume $\langle\psi|$ and $|\varphi\rangle$ are solutions of the following (direct) equations

$$\begin{aligned} z_\nu \langle\chi| &= \langle\chi|(\rho - \nu H), \\ -i \langle\chi_t| &= \frac{1}{\nu} \langle\chi|h(\rho), \end{aligned}$$

and $|\varphi\rangle$ $|\varphi\rangle$ is for conjugate pair Here ρ , H are operators left-acting on a "bra" vectors $\langle\psi|$ associated with an element of a Hilbert space and transforms (bDT) complex numbers λ , z_λ are independent of t . and $\langle\psi_1|$, ρ_1 , $h(\rho)_1$ are defined by

$$\langle\psi_1| = \langle\psi| \left(1 + \frac{\nu - \mu}{\mu - \lambda} P\right), \quad (1.5)$$

$$\rho_1 = T \rho T^{-1}, h_1(\rho) = T h(\rho) T^{-1}, \quad T = \left(1 + \frac{\mu - \nu}{\nu} P\right), \quad (1.6)$$

where $P = |\varphi\rangle\langle\chi| |\varphi\rangle\langle\chi|$. Then the pairs are covariant:

$$z_\lambda \langle\psi_1| = \langle\psi_1|(\rho_1 - \lambda H), \quad -i \langle\dot{\psi}_1| = \frac{1}{\lambda} \langle\psi_1|h_1(\rho).$$

The Lax-pair representation and Darboux covariance properties of this equation have been established in [20]. The cases $f(\rho) = i\rho^3$ and $f(\rho) = i\rho^{-1}$ were considered in [9]. A step to further generalizations for essentially non-Abelian functions, e.g. $h(X) = XA + AX$, $[A, X] \neq 0$, is studied in [17]. The case is the development of the matrix representation of the Euler top model [21].

This example of the theory is more close to the spirit of the sec. 2.4 and more achievements are demonstrated in [22], where abundant set of integrable equations is listed. The list is in a partial correspondence with [9], and give the usual for the DT technique link to solutions via the iteration procedures or dressing chains. The results show how the "true" non-Abelian functions appear in the context of the covariance conditions application. On the way of this specification we use the notion that is similar to one for automorphic function.

2 Covariance principle equations

2.1 One-field Lax pair for Abelian case. Covariance equations

First we would reproduce the "Abelian" scheme, generalizing the study of the example of the Boussinesq equation [12]. To start with the search we should fix the number of fields. Let us consider the third order operator (1.1) with coefficients $b_k, k = 0, 1, 2, 3$, reserving a_k for the second operator in a Lax pair. Suppose, both operators depend on the only potential function w . We would restrict ourselves by the case of $b_{ix} = 0$ and the special choice $b_3 = 1$, $b_1 = b(w)$, $b_0 = G(w)$. The problem we consider now may be formulated as follows: To find restrictions on the coefficients $b_2(t), b(w, t), G(w)$ compatible with DT transformations rules of the potential function w induced by DT for b_i . The standard DT (4.45) for the third order operator coefficients (Matveev generalization [1]) yields (we denote $\partial f = f'$)

$$b_2[1] = b_2 + b'_3, \quad (2.7)$$

$$b_1[1] = b_1 + b'_2 + 3b_3\sigma', \quad (2.8)$$

$$b_0[1] = b_0 + b'_1 + \sigma b'_2 + 3b_3(\sigma\sigma' + \sigma''), \quad (2.9)$$

having in mind that the "elder" coefficient b_3 does not transform. Note also, that $b'_3 = 0$ yields invariance of the coefficient b_2 .

The general idea of DT form-invariance may be realized considering the coefficients transforms to be consistent with respect to the fixed transform of w . Generalizing the analysis of the third order operator transformation [12], one arrives at the equations for the functions $b_2(t), b(w, t), G(w)$. The covariance of the spectral equation

$$b_3\psi_{xxx} + b_2(t)\psi_{xx} + b(w, t)\psi_x + G(w)\psi = \lambda\psi \quad (2.10)$$

may be considered separately, that leads to the link between b_i only. We study the problem of the (2.10) in the context of Lax representation for some nonlinear equation, hence the covariance of the second Lax equation is taken into account from the very beginning. We name such principle as the "*principle of joint covariance*" [15]. The second (evolution) equation of the case is :

$$\psi_t = a_2(t)\psi_{xx} + a_1(t)\psi_x + w\psi, \quad (2.11)$$

with the operator in the r.h.s. having again the form of (1.1). We do not consider here a dependence of a_i, b_i on x for the sake of brevity, leaving this interesting question to the next paper.

If one consider the L and A operators of the form (1), specified in equations (2.10) and (2.11). as the Lax pair equations, the DT of w implied by the covariance of (2.11), should be compatible with DT formulas of both coefficients of (2.10) depending on the only variable w .

$$\begin{aligned} a_2[1] &= a_2 = a(x, t), \\ a_1[1] &= a_1(x, t) + Da(x, t) \\ a_0[1] &= w[1] = w + a'_1 + 2a_2\sigma' + \sigma a'_2 \end{aligned} \quad (2.12)$$

Next important relations being in fact the identities in the DT transformation theory [6], see the introduction, are the particular cases of the generalized Miura map, (1.4):

$$\sigma_t = [a_2(\sigma^2 + \sigma_x) + a_1\sigma + w]_x \quad (2.13)$$

for the problem (2.11) and, for the (2.10)

$$\sigma^3 + 3\sigma_x\sigma + \sigma_{xx} + b(w, t)\sigma + G(w) = \text{const}; \quad (2.14)$$

ϕ is a solution of both Lax equations. Suppose now that the coefficients of the operators are analytical functions of w together with its derivatives (or integrals) with respect to x (such functions are named functions on prolonged space [26]). For the coefficient G it means

$$G = G(\partial^{-1}w, w, w_x, \dots, \partial^{-1}w_t, w_t, w_{tx}, \dots), \quad (2.15)$$

The covariance condition is obtained for the Frechê derivative (FD) of the function G on the prolonged space, or the first terms of Taylor series for (2.15), read

$$G(w + a'_1 + 2a_2\sigma' + \sigma a'_2) = G(w) + G_{w_x}(a'_1 + 2a_2\sigma' + \sigma a'_2)' + G_{\partial^{-1}w_t}(a_{1t} + 2a_2\partial^{-1}(\sigma'_t) + \partial^{-1}(\sigma)) + \dots, \quad (2.16)$$

where we show only terms of further importance; the expression simplifies if a_2 does not depend on x . Quite similar condition one have for the $b_1 = b(w, t)$, with which we would start. In the analogy with the expressions (2.16, 2.8) one obtains

$$b'_2 + 3b_3\sigma' = b_w(a'_1 + 2a_2\sigma' + \sigma a'_2) + b_{w'}(a'_1 + 2a_2\sigma' + \sigma a'_2)' \dots \quad (2.17)$$

This equation we name the (first) "*joint covariance equation*" that guarantee the consistency between transformations of the coefficients of the Lax pair (2.11), (2.10). In the frame of our choice $a'_2 = 0$, the equation simplifies

$$\begin{aligned} 3b_3 &= 2b_w a_2, \\ b'_2 &= b_w a'_1, \end{aligned} \quad (2.18)$$

or

$$\begin{aligned} b_w &= 3b_3/2a_2 \\ b'_2 &= a'_1 3b_3/2a_2. \end{aligned} \quad (2.19)$$

So, if one wants to save the form of the standard DT for the variable w (potential) the simple comparison of both transformation formulas gives for $b(w)$ the following connection (with arbitrary function)

$$b(w, t) = 3w/2 + \alpha(t). \quad (2.20)$$

Equalizing the expansion (2.16) with the transform of the $b_0 = G(w)$ yields:

$$b'_1 + \sigma b'_2 + 3b_3(\sigma^2/2 + \sigma')' = G_{w_x}(a'_1 + 2a_2\sigma' + \sigma a'_2)' + G_{\partial^{-1}w_t}[a_{1t} + 2\partial^{-1}(a_2\sigma'_t) + \partial^{-1}(\sigma a'_2)_t] + \dots \quad (2.21)$$

This second "*joint covariance equation*" also simplifies when $a'_2 = 0$:

$$3b_3w'/2a_2 + 3b_3(\partial^{-1}\sigma_t - w)'/a_2 + 3b_3\sigma''/2 = G_{w_x}(2a_2\sigma')' + G_{\partial^{-1}w_t}[2a_2\sigma_t] + \dots \quad (2.22)$$

when (2.20) is accounted. Note, that the "Miura" (2.11) linearizes the FD with respect to σ . Finally,

$$\begin{aligned} G_{w_x} &= 3b_3/2a_2 \\ G_{\partial^{-1}w_t} &= 3b_3/2a_2^2 \end{aligned} \quad (2.23)$$

Compare with the formula $w[1] = w + 2\sigma_x$. The transformation for the potential w follows from the last equation of the system (2.9), i.e.,

$$G[1] = G + 3w_x/2 + 3(\sigma^2/2 + \sigma_x)_x. \quad (2.24)$$

Such equation determine the functional dependence of $G(u)$ and we would name such equations as *joint covariance equations*. We see that further analysis is necessary due to such constraint (reduction) existence. Then, similar to the case of KdV equation, the covariant equations (2.10, 2.11)

$$a_{1t} + a_2 a''_1 + a_1 a'_1 = 0, \quad (2.25)$$

which get the form of the Burgers equation after (2.19) account. Finally the "lower" coefficient of the third order operator is expressed by

$$G(w, t) = 3b_3 w_x / 2a_2 + 3b_3 a_1' \partial^{-1} w / 2(a_2)^2 + 3b_3 \partial^{-1} w_t / 2a_2^2. \quad (2.26)$$

Statement 1 The expressions (2.11, 2.10, 2.20, 2.26) define the covariant Lax pair when the constraints (2.25, 2.19) are valid.

Such equation for (2.11) compare with Riccati equation (stationary version) for the second order spectral problem corresponding to KdV provided by the name of Miura equation. If one would use the equation (3.30) in (2.24), the time-derivative of w appear. Moreover, the further analysis shows that the case we study need to widen the functional dependence in u , namely we should include not only derivatives of w with respect to x , but integrals (inverse derivatives) as arguments of the potential. Let us introduce analytical function G denoting the coefficient b_0 .

The DT transform of G after substitution of (2.24) gives

$$G(w) + 3w_x/2 + 3(-\sigma_t - w_x)/2 + 3\sigma_{xx}/2. \quad (2.27)$$

Equalizing (2.27) and (2.24) yields

$$G_{w_x} = 3/4; G_{\partial^{-1} w_t} = -3/4.$$

That leads to the exact form of the Lax pair for the Boussinesq equation from [7] for the choice of $\alpha = -3/4$.

Remark 1. We cut the Frechê differential formulas on the level that is necessary for the minimal flows. The account of higher terms leads to the whole hierarchy [?].

Remark 2. We cut the Frechê differential formulas on the level that is necessary for the minimal flows. The account of higher terms lead to higher flows (higher KdV, for example) [24].

3 The solitons of the generalized Boussinesq and dressing chain for the Boussinesq equation

To produce a simplest soliton solution of the generalized Boussinesq equation, it is enough to start from zero potential in the Lax pair equations (2.10, 2.11).

$$\begin{aligned} b_3 \psi_{xxx} + b_2(t) \psi_{xx} + \alpha(t) \psi_x &= \lambda \psi \\ \psi_t &= a_2(t) \psi_{xx} + a_1(t) \psi_x \end{aligned} \quad (3.28)$$

The seed solution should satisfy both Lax equations (3.28) The dressing formula for the zero seed potential (4.45) is standard and includes this only function.

$$w_s = a_1' + 2a_2 \sigma' + \sigma a_2' = a_1' + 2a_2 \log_{xx} \phi(x, t) \quad (3.29)$$

Going to the dressing chain, we use the method from [12]. We would restrict ourselves to the case of constant $b_2 = 0$, $b_3 = 1$, $b_1 = b$, $b_0 = u$. The general construction is quite similar.

the covariant equations (2.11, 2.10) are accompanied by the following equation

$$\sigma_t = -(\sigma^2 + \sigma_x)_x - w_x \quad (3.30)$$

for the problem (2.11) and

$$\sigma^3 + 3\sigma_x\sigma + \sigma_{xx} + b\sigma + G = \text{const}, \quad (3.31)$$

for (2.10), see (2.13), compare with (2.14) that was Riccati equation (stationary version) for the second order spectral problem corresponding to KdV. If one would use the equation (3.30) in (2.24), the time-derivative of w appear.

Namely the "Miura" equations (3.30, 3.31 together with the DT formula 1.5

$$w_{n+1} = w_n + 2l_{nx}\sigma_n \quad (3.32)$$

form the basis to produce the DT dressing chain equations.

We express the iterated potential w_n from 3.30

$$-\sigma_{nt} - (\sigma_n^2 + \sigma_{nx})_x = w_{nx} \quad (3.33)$$

and substitute it into 3.32. The first dressing chain equation

$$\sigma_{n+1,t} - \sigma_{nt} = (\sigma_{n+1}^2 + \sigma'_{n+1})' - (\sigma_n^2 - \sigma'_n)'. \quad (3.34)$$

Next chain equation is obtained when one plugs the potential from (3.33) to the iterated (??)

$$\sigma_n^3 + 3\sigma'_n\sigma_n + \sigma''_n + (-3u_n/2 + \alpha)\sigma_n - 3u'_n/4 + 3\partial^{-1}u_{nt} = c_n. \quad (3.35)$$

4 Non-Abelian case. Zakharov-Shabat (ZS) problem.

4.1 Compatibility condition.

In the case $a'_2 = 0$ by which we have restricted ourselves, the Lax system (2.11,2.10) produces the following compatibility conditions:

$$\begin{aligned} 2a_2b'_3 &= 3b_3a'_2, \\ b_{3t} &= a_2b''_3 + 2a_2b'_2 + a_1b'_3 - 3b_3a''_2 - 3b_3a'_1 - 2b_2a'_2 \\ b_{2t} &= a_2b''_2 + 2a_2b'_1 + a_1b'_2 - b_3a'''_2 - b_2a''_2 - b_1a'_2 - 3b_3a''_1 - 2b_2a'_1 - 3b_3a'_0 \\ b_{1t} &= a_2b''_1 + a_1b'_1 - b_3a'''_1 - b_2a''_1 - b_1a'_1 - 3b_3a''_0 - 2b_2a'_0 + 2a_2b'_0 \\ b_{0t} &= a_1b'_0 - b_1a'_0 + a_2b''_0 - b_2a''_0 - b_3a'''_0 \end{aligned} \quad (4.36)$$

In the particular case of $a'_2 = 0$ we have at once the direct corollary of the first of equalities (4.36) $b'_3 = 0$; in the rest of the equations the restriction (4.36) is taken into account.

4.2 Compatibility conditions for two general ZS problems

Let us list equations that appear as compatibility condition of two first order operators with coefficients to form a nonabelian set. We recall these equation just to examine the complete set of them from the DT invariance point of view. The pair we study is

$$\psi_t = (a_0 + a_1D)\psi, \quad (4.37)$$

$$\psi_y = (b_0 + b_1D)\psi, \quad (4.38)$$

where a_i, b_i are functions of x, y, t and let D be a differentiation with respect to x operator. The equations of compatibility are

$$[b_1, a_1 = 0] \quad (4.39)$$

$$a_{1y} - b_{1t} + [a_0, b_1] + [a_1, b_0] + a_1 b'_1 - b_1 a'_1 = 0. \quad (4.40)$$

$$a_{0y} - b_{0t} + [a_0, b_0] + a_1 b'_0 - b_1 a'_0 = 0. \quad (4.41)$$

The "''" denotes the derivative with respect to x .

If the coefficients a_1, b_1 do not depend on x, y, t , the first two equations simplify.

$$[b_1, a_1 = 0] \quad (4.42)$$

$$[a_0, b_1] + [a_1, b_0] = 0. \quad (4.43)$$

Analyzing the conditions, one sees, that the direct proof of the heredity of (4.40) is a corollary of Jacoby identity.

If one changes the differentiation operator D in (??) to the shift operator T , the equations are changing as follows

In the next generalization [10], the operator T can be considered as automorphism in a differential ring. Some direct generalization of (??) is achieved, if the operator D is considered as abstract differentiation. See. for example, [15] where D is a commutator.

4.3 Covariance equations

Let us change notations

$$\psi_t = (J + uD)\psi, \quad (4.44)$$

where the operator J does not depend on x, y, t and the potential $a_0 = u = u(x, y, t)$ is a function of all variables. The operator D now is a differentiation by x . The transformed potential

$$\tilde{u} = u + [J, \sigma], \quad (4.45)$$

the $\sigma = \phi_x \phi^{-1}$ is defined by the same formula as before. The covariance of the operator in (4.37) follows from general transformations of the coefficients of a polynomial [7]. The operator J does not transform.

Suppose the second operator of a Lax pair has the same form, but with different entries and derivatives.

$$\psi_y = (Y + wD)\psi, \quad (4.46)$$

where the potential $w = F(u)$ is a function of the potential of the first (4.44) equation. The principle of joint covariance [15] hence reads:

$$\tilde{w} = w + [Y, \sigma] = F(u + [J, \sigma]), \quad (4.47)$$

with the direct corollary

$$F(u) + [Y, \sigma] = F(u + [J, \sigma]). \quad (4.48)$$

It implies the same functional dependence of the coefficient F on u before and after transformation. So, the equation (4.48) defines the function $F(u)$, we shall name this equation as **joint covariance equation**. In the case of abelian algebra we used the Taylor series (generalized by use of a Frechet derivative) to determine the function. Now some generalization is necessary. Let us make some general remarks.

A class operator-valued function $F(u)$ of an operator u in Banach space may be considered as a generalized Taylor series with coefficients that are expressed in terms of Frechet derivatives [?]. In a sense of the space norm the linear in u part of the series approximate the function

$$F(u) = F(0) + F'(0)u + \dots$$

The representation is not unique and the similar expression

$$F(u) = F(0) + u\hat{F}'(0) + \dots$$

may be introduced (some fundamentals about the definitions are given in Appendix). Both expressions however are not Hermitian, hence not suitable for a majority of physical models. It means, that the class of such operators is too restrictive. To explain what we have in mind, let us consider examples.

4.4 Important example

From a point of view of the physical modelling the following approximation

$$F(u) = H^+u + uH$$

is preferable for the important class of Hermitian operators. The case of the Hermitian H is included. Such models could be applied to quantum theories: introduction of this approximation is similar to "phi in quadro" (Landau-Ginzburg) model [1]. So, let it be

$$F(u) = Hu + uH, \tag{4.49}$$

by a direct calculation in (4.48) one arrives at the equality

$$[Y, \sigma] = H[J, \sigma] + [J, \sigma]H. \tag{4.50}$$

The obvious choice for arbitrary σ is $Y = H^2, J = H$.

The compatibility conditions (?? for (4.44) and (4.46), when σ is $Y = H^2, J = H$ yields

$$\begin{aligned} [J, J^2] &= 0 \\ [J, u]H + \{u[J, H]\} + [u, H]J &= 0 \\ u_y - Hu_t - u_tH + [u, H]u + u[u, H] + JHu_x + Ju_xH + HJu_x &= 0 \end{aligned} \tag{4.51}$$

So this important case of $[H, J]=0$ gives

$$\begin{aligned} [J, u]H + [u, H]J &= 0 \\ u_y - Hu_t - u_tH + [u, H]u + u[u, H] + JHu_x + Ju_xH + HJu_x &= 0 \end{aligned} \tag{4.52}$$

Both conditions are covariant, the first of them is a DT-hereditary constraint, that may be checked directly. We hence obtain a class of DT-integrable equations/Lax pairs.

$$u_y - \{H.u\}_t + [u^2, H] + JHu_x + Ju_xH + HJu_x = 0, \tag{4.53}$$

It is seen that in the case $J=H$, the first of equations (4.52) is valid identically.
if the potential does not depend on t it is reduced to the next equation:

$$u_y + [u^2, J] + J^2 u_x + J u_x J + J^2 u_x = 0, \quad (4.54)$$

and x -independence yields the generalized Euler top equations

$$u_y + [u^2, J] = 0, \quad (4.55)$$

which Lax pair (4.44,4.46) with $Y = J^2$ was found by Manakov [21]. So, we used the compatibility condition to find the form of integrable equation and reduction tracing the simplifications appearing for the subclasses of covariant potentials. While doing this we also check the invariance of the equation and heredity of the constraints.

4.5 Covariant combinations of symmetric polynomials

The next natural example appears if one examine the link (4.50).

$$P_2(H, u) = H^2 u + H u H + u H^2$$

The direct substitution in the covariance and compatibility equations leads to covariant constraint that goes to identity if, $Y = H^3, J = H$.

It is easy to check more general $Y = J^n, J = H$ connection possibility that leads to the covariance of

$$P_n(H, u) = \sum_{p=0}^n H^{n-p} u H^p.$$

Such observation was exhibited in [?]. On the way of a further generalization let us consider

$$f(H, u) = H u + u H + S^2 u + S u S + u S^2 \quad (4.56)$$

Plugging (4.56) as $F(u) = f(H, u)$ into (4.48), representing $Y = AB + CDE$ yields

$$A[B, \sigma] + [A, \sigma]B + CD[E, \sigma] + C[D, \sigma]E + [C, \sigma]DE = H[J, \sigma] + [J, \sigma]H + S^2[J, \sigma] + S[J, \sigma]S + [J, \sigma]S^2.$$

The last expression turns to identity if $A = B = J = H, C = \alpha H, D = \alpha H, S = \beta H$ and $[\alpha, H] = 0, [\beta, H] = 0$ with the link $\alpha^3 = \beta^2$.

Statement *Darboux covariance define a class of homogeneous polynomials $P_n(H, u)$, symmetric with respect to cyclic permutations. A linear combination of such polynomials $\sum_{n=1}^N \beta_n P_n(H, u)$ with the coefficients commuting with u, H is also covariant, if the element $Y = \sum_{n=1}^N \alpha_n H^{n+1}$ and $\alpha_1 = \beta_1 = 1, \alpha_n^{n+2} = \beta^{n+1}, n \neq 1$. A proof could be made by induction that is based on homogeneity of the P_n and linearity of the constraints with respect to u . The functions $F_H(u) = \sum_0^\infty a_n P_{Hn}(u)$ satisfy the constraints if the series converges.*

5 Conclusion

The equation () generalize Boussinesq equation and has direct physical sense for both case of waves launched by initial or boundary problem. In the first case t -dependence of coefficients means the external conditions varied with time. In the second case the T and x coordinates

interchange: the coefficients dependence may be interpreted as conditions that are changed when a wave propagates (e.g. a bottom slope for the surface waves). The main result of this paper is the covariant equation (4.48) or its version (4.50). A class of potentials from [22] contain polynomials $P_{Hn}(u)$ and give alternative expressions for it. The linear combinations, introduced here could better reproduce physical situation of interest.

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6 Appendix. Right and left Frechét derivatives

The notion of a derivative an operator by other one is defined in a Banach space B [?]. Two specific features in the case of a operator-function $F(u) \in B$ $u \in B$ should be taken into account: a norm choice when a limiting procedure is made and the nonabelian character of expressions while the differential and difference introduced.

Definition. Let a Banach space B have a structure of a differential ring. Let F be the operator from B to B' defined on the open set of B . The operator is named the left-differentiable in $u_0 \in B$ if there exist a linear restricted operator $L(u_0)$, acting also from B to B' with the property

$$L(u_0 + h) - L(u_0) = L(u_0)h + \alpha(u_0, h), \|h\| \rightarrow 0, \quad (6.57)$$

where $\|\alpha(u_0, h)\|/\|h\| \rightarrow 0$. The operator $L(u_0) = F'(u_0)$ is referred as the operator of the (strong) left derivative of the function $F(u)$. The right derivative $\hat{F}'(u_0)$ could be defined by the similar expression and conditions, if one changes $Lh \rightarrow h\hat{L}$ in the (??).

The Gatoux derivative